

Using inverses to solve systems

If A^{-1} exists, we can solve $Ax=b$ by multiplying by A^{-1}

$$A^{-1}(Ax) = A^{-1}b$$

$$(A^{-1}A)x = A^{-1}b$$

$$Ix = A^{-1}b$$

$$x = A^{-1}b$$

Hence $Ax=b$ has solution $A^{-1}b$. Moreover, since A has n pivots (as it is invertible), this solution is unique.

Ex: Solve $2x_1 + 4x_2 - 2x_3 = 4$

$$4x_1 + 4x_2 = 8$$

$$x_2 + 3x_3 = 8$$

In other words, solve $Ax=b$, where

$$A = \begin{bmatrix} 2 & 4 & -2 \\ 4 & 4 & 0 \\ 0 & 1 & 3 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 4 \\ 8 \\ 8 \end{bmatrix}.$$

We already know $A^{-1} = \begin{bmatrix} 3/4 & -1/8 & -1 \\ -3/4 & 3/8 & 1 \\ 1/4 & -1/8 & 0 \end{bmatrix}$

$$\text{Hence } x = A^{-1}b = \begin{bmatrix} 3/4 & -1/8 & -1 \\ -3/4 & 3/8 & 1 \\ 1/4 & -1/8 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 8 \end{bmatrix} = \begin{bmatrix} -6 \\ 8 \\ 0 \end{bmatrix}$$

$$\text{or } x = -6, y = 8, z = 0.$$

Properties:

- a) If A is invertible, then its inverse is unique
- b) If $AB=I$, then A is invertible
- c) If A is invertible, then so is A^{-1} and $(A^{-1})^{-1}=A$
- d) Let A, B be invertible $n \times n$ matrices.
Then AB is invertible and $(AB)^{-1}=B^{-1}A^{-1}$

Proof:

- a) Let B and C be inverses of A .

$$\text{Then } B = BI = B(AC) = (BA)C = IC = C$$

Hence $B=C$ and so there is only one inverse

- b) Since $AB=I$, $(AB)x=b$ has a solution (i.e. $x=b$) $\forall b$.
Setting $Bx=y$, we have $Ay=b$ has a solution (i.e. $y=Bb$) $\forall b$
Thus $[A|b]$ cannot have pivot in last column for all b
Hence A must have n pivots $\Rightarrow A$ is invertible. ▀

- c) Since $A^{-1} \cdot A = I$, A is the inverse of A^{-1} (i.e. $(A^{-1})^{-1}=A$)

- d) Since $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$

$B^{-1}A^{-1}$ is the inverse of AB

that is, $(AB)^{-1} = B^{-1}A^{-1}$

Transposes:

If A is an $m \times n$ matrix, then the transpose of A is the $n \times m$ matrix, denoted by A^T obtained by flipping the rows and columns of A .

$$\text{Ex: } A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 5 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 3 \\ 0 & -1 \\ 2 & 5 \end{bmatrix}$$

The transpose of a vector $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ is called a row vector
 $v^T = [v_1 \cdots v_n]$

Notice: If $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ and $w = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$, then

Dot Product $v \cdot w = v_1 w_1 + \cdots + v_n w_n = v^T w$ matrix multiplication

Connection to Linear Transformations:

If $S: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation with associated matrix A then the linear transformation associated to A^T is called the dual of S , denoted S^*

Understanding this goes beyond the scope of this course, but feel free to ask me about it in office hours!

Properties: Let A and B have the appropriate sizes.

a) $(A^T)^T = A$

b) $(A+B)^T = A^T + B^T$

c) $(cA)^T = cA^T$

d) $(AB)^T = B^T A^T$

e) $(A^T)^{-1} = (A^{-1})^T$

Proof:

a) Let $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \dots & \dots & a_{mn} \end{bmatrix}.$

Then $A^T = \begin{bmatrix} a_{11} & \dots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \dots & a_{mn} \end{bmatrix}$ and so

$$(A^T)^T = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} = A.$$

b), c) Similar

d) Recitation

e) Recitation



A symmetric matrix is a matrix A satisfying $A^T = A$
(so $a_{ij} = a_{ji} \forall i, j$)

Ex: $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 5 \\ 0 & 5 & 3 \end{bmatrix}$ is symmetric since $A^T = A$

Ex: Let A be any $m \times n$ matrix. Then $A^T A$ is symmetric since $(A^T A)^T = A^T (A^T)^T = A^T A$

Next time: LU Factorization

Given a square matrix A , factor it as $A = LU$,
where L = lower triangular and U = upper triangular

This helps us more efficiently solve systems
 $Ax = b$ when b varies